Journal of Statistical Physics, Vol. 114, Nos. 1/2, January 2004 (© 2004)

The Virial Expansion of a Dilute Bose Gas in Two Dimensions

Hai-cang Ren[1]

Received October 23, 2002; accepted July 11, 2003

In terms of the *s*-wave phase shift of the two-body scattering at thermal wavelength, a systematic perturbative expansion of the Virial coefficients is developed for a two-dimensional dilute system of bosons in its gaseous phase at low temperature. The thermodynamic functions are calculated to the second order of the expansion parameter. The observability of the universal low energy limit of the two dimensional phase shift with a quasi-two dimensional atomic gas in an anisotropic trap is discussed.

KEY WORDS: Viral expansion; dilute Bose gas; two dimensions; renormalized potential; 2D scattering length.

1. INTRODUCTION

The recent advent of laser cooling and atomic trapping techniques makes the physics of a dilute quantum gas experimentally accessible, which led to the observation of the Bose–Einstein condensation of the trapped metallic atoms in three dimensions.[(1)] Theoretical interests on these many-body quantum mechanical system has also revived since then. While the Virial expansion was investigated for a three dimensional dilute gas in both gaseous phase and condensate phase long ago,[(2–4)] a parallel formulation in two dimensions remains to be developed. There are many elegant works on 2D bosons concerning the quasi-Bose condensate near the absolute zero.[(5–10)] The Virial expansion developed in this paper is complementary. As the quasi-two dimensional gas of trapped atoms is also experimentally feasible, the result reported in this paper may be brought to a direct comparison with the measurements.

[1] Physics Department, The Rockefeller University, 1230 York Avenue, New York, New York 10021-6399; e-mail: ren@theory.rockefeller.edu

0022-4715/04/0100-0481/0

As is well-known, a perturbative treatment of a dilute Bose gas in two dimensions suffers from two difficulties: (1) The scattering amplitude vanishes in the zero energy limit and the Born expansion breaks down for a large number of potentials.[11, 12] (2) The long range order parameter corresponding to the Bose condensate ceases to exist at nonzero temperatures because of the fluctuation of the condensate phase.[13] Both difficulties stems from the two dimensional character of the density of states at low energies. We shall focus on the first difficulty in this paper.

The Hamiltonian of a dilute system of interacting bosons is given by

$$H = \sum_{\vec{p}} (p^2 - \mu)\, b_{\vec{p}}^{\dagger} b_{\vec{p}} + \frac{1}{2\Omega} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}_1', \vec{p}_2'} \delta_{\vec{p}_1 + \vec{p}_2, \vec{p}_1' + \vec{p}_2'} \langle \vec{p}_1' \vec{p}_2' | V | \vec{p}_1 \vec{p}_2 \rangle\, b_{\vec{p}_2'}^{\dagger} b_{\vec{p}_1'}^{\dagger} b_{\vec{p}_1} b_{\vec{p}_2} \tag{1}$$

with $b_{\vec{p}}$, $b_{\vec{p}}^{\dagger}$ the annihilation and creation operators of the bosons in their momentum space and μ the chemical potential. We have chosen the unit of mass such that the mass of a boson is $\frac{1}{2}$. In the rest of this paper, two-body potential V is assumed to be isotropic, repulsive and of short range in coordinate space.

The physics of the system depends on the relations among three length scales, the range of the interaction, r_0, the average inter-particle distance $1/\sqrt{n}$ and the thermal wavelength,

$$\lambda = \sqrt{4\pi\beta} \tag{2}$$

with $\beta = (k_B T)^{-1}$. The diluteness is measured by $nr_0^2 \ll 1$ and the classical limit corresponds to $n\lambda^2 \to 0$. The Virial expansion we shall derive applies to the temperature such that $n\lambda^2 \sim 1$ when quantum coherence becomes significant. The effective expansion parameter is the s-wave scattering phase shift at the thermal wavelength, which is $O(\frac{1}{\ln(\lambda^2 r_0^2)^{-1}})$. Therefore, the interaction corrections are far more significant than that of a three dimensional Bose gas at the same diluteness.

In the next section, we shall introduce a renormalized two body potential as a expansion parameter in the dilute limit, which set up the systematics of the perturbation. The corresponding Virial expansion to the second order of the renormalized potential will be developed in Section 3 with typical thermodynamic quantities calculated in Section 4 to the same order. The convergence of the Virial expansion and its relation to the quasi Bose–Einstein condensation will be discussed in the final section.

2. THE RENORMALIZED POTENTIAL

The requirement of renormalizing the interaction potential is not unfamiliar in three dimensions with a hard sphere potential or more realistic

Lennard-Jones potential, for which a straight forward perturbation series breaks down because of the singular behavior of the potential at short distance. The natural choice for the renormalized potential is the exact two-body scattering amplitude at zero energy (scattering length). What is lacking in two dimensions is such a natural choice, since the scattering amplitude vanishes at zero energy. This is analogous to perturbation theory of the quantum chromodynamics, where the infrared slavery and the asymptotic freedom deprive us of a natural scale of the ultraviolet renormalization. A running coupling constant defined at relevant energy scale has to be introduced as the expansion parameter. For a two dimensional Bose gas, we need also to introduce a running coupling constant, which turns out to be the s-wave phase shift at the thermal wavelength of the bosons.

To replace $\langle \vec{p}_1' \vec{p}_2' | V | \vec{p}_1 \vec{p}_2 \rangle$ with appropriate renormalized quantity at low energies, we focus on the two body sector of the Hamiltonian (1) and define

$$\mathscr{V} = \tfrac{1}{2}\left(V e^{-\beta H} e^{\beta H_0} + e^{\beta H_0} e^{-\beta H} V\right). \tag{3}$$

The hermitian operator $\mathscr{V}$ can be formally expanded according to the power of the potential V and vice versa. To the leading order $\mathscr{V} = V$ and it is straight forward to show that

$$V = \mathscr{V} + \tfrac{1}{2}\left[\int_0^\beta d\tau\, \mathscr{V} e^{-\tau H_0} \mathscr{V} e^{\tau H_0} + \int_0^\beta d\tau\, e^{\tau H_0} \mathscr{V} e^{-\tau H_0} \mathscr{V}\right] + O(\mathscr{V}^3). \tag{4}$$

Sandwiching $\mathscr{V}$ between two-body states and completing the integral over τ, we find that

$$\begin{aligned} \langle \vec{p}_1' \vec{p}_2' | V | \vec{p}_1 \vec{p}_2 \rangle &= \langle \vec{p}_1' \vec{p}_2' | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle \\ &\quad + \frac{1}{2} \sum_{\vec{k}_2, \vec{k}_2} \left[\frac{e^{\beta(p_1^2 + p_2^2 - k_1^2 - k_2^2)} - 1}{p_1^2 + p_2^2 - k_1^2 - k_2^2} + \frac{e^{\beta(p_1'^2 + p_2'^2 - k_1^2 - k_2^2)} - 1}{p_1'^2 + p_2'^2 - k_1^2 - k_2^2} \right] \\ &\quad \times \langle \vec{p}_1' \vec{p}_2' | \mathscr{V} | \vec{k}_1 \vec{k}_2 \rangle \langle \vec{k}_1 \vec{k}_2 | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle + O(\mathscr{V}^3). \end{aligned} \tag{5}$$

As we shall see, the matrix element $\langle \vec{p}_1' \vec{p}_2' | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle$ vanishes in the limit $\beta \to \infty$ and is a proper choice of the renormalized potential at low temperature. Furthermore, replacing V by its formal expansion in $\mathscr{V}$, the systematics to all orders is restored. In terms of the two-body matrix element of $\mathscr{V}$, the Hamiltonian of the system, (1) becomes

$$H = \sum_{\vec{p}} (p^2 - \mu)\, b^\dagger_{\vec{p}} b_{\vec{p}} + \frac{1}{2\Omega} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}'_1, \vec{p}'_2} \delta_{\vec{p}_1+\vec{p}_2, \vec{p}'_1+\vec{p}'_2} \langle \vec{p}'_1 \vec{p}'_2 | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle\, b^\dagger_{\vec{p}'_2} b^\dagger_{\vec{p}'_1} b_{\vec{p}_1} b_{\vec{p}_2}$$

$$+ \frac{1}{2\Omega} \sum_{\vec{p}_1, \vec{p}_2, \vec{p}'_1, \vec{p}'_2} \delta_{\vec{p}_1+\vec{p}_2, \vec{p}'_1+\vec{p}'_2} \langle \vec{p}'_1 \vec{p}'_2 | \mathscr{V} - V | \vec{p}_1 \vec{p}_2 \rangle\, b^\dagger_{\vec{p}'_2} b^\dagger_{\vec{p}'_1} b_{\vec{p}_1} b_{\vec{p}_2}, \tag{6}$$

where the last term is understood as an power series of $\mathscr{V}$ starting from the order $\mathscr{V}^2$, like the renormalization counter terms in relativistic field theories. A similar method of renormalization was developed in the context of a lattice gas with on-site exclusion.[14]

The matrix element (3) is related to the binary kernel of Lee and Yang[3] through

$$\langle \vec{p}'_1 \vec{p}'_2 | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle = -\tfrac{1}{2} [\langle \vec{p}'_1 \vec{p}'_2 | B(\beta) | \vec{p}_1 \vec{p}_2 \rangle\, e^{\beta(p_1^2 + p_2^2)}$$
$$+ \langle \vec{p}_1 \vec{p}_2 | B(\beta) | \vec{p}'_1 \vec{p}'_2 \rangle^*\, e^{\beta(p_1'^2 + p_2'^2)}], \tag{7}$$

and the Virial expansion developed here for a 2D Bose gas is parallel to that of Lee and Yang for a 3D hard sphere gas.

In terms of the total momentum $\vec{P} = \vec{p}_1 + \vec{p}_2$, $\vec{P}' = \vec{p}'_1 + \vec{p}'_2$, and the relative momentum $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$, $\vec{p}' = \frac{1}{2}(\vec{p}'_1 - \vec{p}'_2)$, we have

$$\langle \vec{p}'_1 \vec{p}'_2 | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle = \tfrac{1}{2} \delta_{\vec{P}', \vec{P}} [e^{2\beta p^2} \langle \vec{p}' | V e^{-\beta h} | \vec{p} \rangle + e^{2\beta p'^2} \langle \vec{p}' | e^{-\beta h} V | \vec{p} \rangle], \tag{8}$$

where

$$h = -2\nabla_r^2 + V(\vec{r}) \tag{9}$$

is the Hamiltonian of a potential scattering problem with $\vec{r}$ the relative coordinate. Inserting the complete set of eigenstates of h, i.e., $h\,|n\rangle = E_n\,|n\rangle$, we have

$$\langle \vec{p}' | V e^{-\beta h} | \vec{p} \rangle = \sum_n e^{-\beta E_n} \langle \vec{p}' | V | n \rangle \langle n | \vec{p} \rangle \tag{10}$$

and $\langle \vec{p}' | e^{-\beta h} V | \vec{p} \rangle = \langle \vec{p} | V e^{-\beta h} | \vec{p}' \rangle^*$. For an isotropic potential, $|n\rangle$ is specified by the azimuthal quantum number, m, and the radial momentum, k, with $E = 2k^2$. The corresponding wave function

$$\langle \vec{r} \,|\, k, m \rangle = N_{km} \frac{1}{\sqrt{2\pi}} e^{im\phi} u_m(k \,|\, r) \tag{11}$$

where the radial wave function $u_m(k \mid r)$ approaches asymptotically

$$u_m(r \mid k) \simeq \sqrt{\frac{2}{\pi k r}} \cos\left[kr - \frac{m\pi}{2} - \frac{\pi}{4} + \delta_m(k) \right] \tag{12}$$

for $kr \gg 1$ with $\delta_m(k)$ the phase shift and N_{km} is a normalization constant. The partial-wave expansion of (10) reads

$$\langle \vec{p}' \mid V e^{-\beta h} \mid \vec{p} \rangle = \frac{1}{\Omega} V_0(p', p) + \frac{2}{\Omega} \sum_{m=1}^{\infty} V_m(p', p) \cos m\Phi, \tag{13}$$

where

$$V_m(p', p) = 2\pi \sum_k N_{km}^2 e^{-2\beta k^2} \int_0^\infty dr'\, r' J_m(p'r')\, V(r')\, u_m(k \mid r')$$
$$\times \int_0^\infty dr\, r u_m(k \mid r)\, J_m(pr) \tag{14}$$

with Φ the angle between $\vec{p}$ and $\vec{p}'$, and $J_m(z)$ the Bessel function. The infinite volume limit of the integral over r and the sum over k in (14) have to be evaluated carefully. We defer the details to Appendix A and quote only the result here,

$$V_m(p', p) = -4e^{-2\beta p^2} \left(\frac{p'}{p} \right)^m \sin 2\delta_m(p) + \pi \mathscr{P} \int_0^\infty dk\, k e^{-2\beta k^2} \frac{v_m(p', k)\, v_m(p, k)}{k^2 - p^2} \tag{15}$$

with

$$v_m(p, k) = \int_0^\infty dr\, r J_m(pr)\, V(r)\, u_m(k \mid r), \tag{16}$$

$$v_m(p, p) = -\frac{4}{\pi} \sin \delta_m(p), \tag{17}$$

and $\mathscr{P}$ the principal value of the integral. For a short-range potential, the wave function $u_m(k \mid r)$ normalized according to (12) takes the form

$$u_m(k \mid r) \simeq \frac{\sin \delta_m(k)}{k^{2m}} f_m(r) \tag{18}$$

in the limit $kr \to 0$, with $f_m(r)$ independent of k. The phase shifts display the following low energy behavior

$$\delta_0(k) \simeq \frac{\pi}{2 \ln ka} \tag{19}$$

and

$$\delta_m(k) \sim k^{2m}, \tag{20}$$

for $m \neq 0$ where a is the s-wave scattering length in two dimensions. For a hard sphere potential, $a = \frac{1}{2} e^{\gamma} r_0$ with r_0 the radius of the sphere and γ the Euler constant. But the scattering length and the range of the potential may not be comparable in general. The relevant length for the Virial expansion developed in this paper is the scattering length a. It follows from (17) and (18) that

$$v_m(p, k) = -\frac{4}{\pi} \left(\frac{p}{k} \right)^m \sin \delta_m(k) \tag{21}$$

for k and p both small, and the low energy and low temperature approximation of (15) reads

$$V_m(p', p) \simeq -4e^{-2\beta p^2} \left(\frac{p'}{p} \right)^m \sin 2\delta_m(p) + \frac{16}{\pi} p'^m p^m \mathscr{P} \int_0^\infty dk\, k^{-2m+1} e^{-2\beta k^2} \frac{\sin^2 \delta_m(k)}{k^2 - p^2}. \tag{22}$$

Equation (19) was proved rigorously by Chan *et al.*[12] for a general class of potentials that fall off faster than $\frac{1}{r^2 \ln r}$ for $r \to \infty$. They also proved that for the same class of potentials, the correction to the corresponding function $f_0(r)$ of is of the order of k^2. As this involves only the long wavelength limit of the scattering, their conclusion can also generalized to the repulsive potential that becomes singular as $r \to 0$. Therefore, all the logarithmic dependence on the range of the interaction of, $\ln a$, is absorbed in the s-wave phase shift through (22). This way the s-wave channel dominates over all other partial wave channels to all orders of a low energy expansion, different from three dimensions. The universality found in ref. 12 is highlighted in the low temperature thermodynamics of the 2D system. In what follows, we shall suppress the subscript "0" for the s-wave and introduce a running coupling constant at the scale of the thermal wavelength,

$$\alpha \equiv \frac{1}{\ln \frac{\lambda^2}{2\pi a^2} - \gamma}, \tag{23}$$

we have $\alpha \simeq \frac{1}{\pi}\delta(\frac{\sqrt{2\pi}\, e^{\gamma/2}}{\lambda})$ and the s-wave phase shift becomes

$$\delta = -\pi\alpha + \pi\alpha^2 \left(\ln \frac{k^2\lambda^2}{2\pi} + \gamma \right) + \cdots. \tag{24}$$

As we shall see, the choice of the constant pertaining to the logarithm of (23) is to make the second Virial coefficient free from $O(\alpha^2)$ corrections. It follows from (10), (15), and (24) that the renormalized potential at low energies reads

$$\langle \vec{p}_1' \vec{p}_2' | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle = \frac{\delta_{\vec{P}, \vec{P}'}}{\Omega} \left[8\pi\alpha - 8\pi\alpha^2 \left(\ln \frac{pp'\lambda^2}{2\pi} + \gamma \right) \right. \\ \left. + 16\pi\alpha^2 \mathscr{P} \int_0^\infty dk\, k \frac{e^{2\beta(p^2 - k^2)}}{k^2 - p^2} + \cdots \right], \tag{25}$$

and this expression will be applied extensively in the subsequent sections.

The dependence of the dimensionless running coupling constant (23) on T and a is completely analogous to that of a relativistic field theory of zero masses in 3D with T corresponding to the renormalization energy scale and a the ultraviolet cutoff.

3. VIRIAL EXPANSION

The thermodynamics of a uniform gas is determined completely by its equation of state, usually expressed in the form of the Virial expansion,

$$\frac{p}{k_B T} = \sum_{l=1}^{\infty} b_l z^l \tag{26}$$

and

$$n = \frac{\partial}{\partial \ln z} \left(\frac{p}{k_B T} \right)_T \tag{27}$$

with n the number density and $z = e^{\beta\mu}$ the fugacity. The lth Virial coefficient, b_l is determined by the quantum mechanics of l particles, the exact solution of which is in general unavailable for $l > 2$. For an ideal Bose gas in two dimensions, $b_l = \frac{1}{\lambda^2 l^2}$. On writing

$$b_l = \frac{1}{\lambda^2 l^2} + b'_l, \tag{28}$$

we have

$$\frac{p}{k_B T} = \frac{1}{\lambda^2} g_2(z) + \Gamma \tag{29}$$

with $g_2(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^2}$ and $\Gamma \equiv \sum_{l=1}^{\infty} b'_l z^l$. The perturbative expansion of Γ is represented by thermal diagrams.

The thermal diagrams of Γ to the third order in α is shown in Fig. 1, where a solid line represents a boson propagator, $\frac{i}{i\omega_n - p^2 + \mu}$ with ω_n the Matsubara energy of the boson, a solid circle vertex is associated to the factor $\langle \vec{p}'_1 \vec{p}'_2 | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle$ with $\vec{p}_1, \vec{p}_2$ $(\vec{p}'_1, \vec{p}'_2)$ the incoming (outgoing) momenta and an open circle vertex denotes $\langle \vec{p}'_1 \vec{p}'_2 | V - \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle$, the analog of the renormalization counter term in a relativistic field theory. In what follows, we shall calculate the Γ to the order α^2, which is one order beyond the mean field approximation.

On writing

$$\Gamma = \Gamma_a + \Gamma_b + \Gamma_c + \Gamma_d + \cdots \tag{30}$$

with Γ_a, Γ_b, Γ_c, and Γ_d standing for the contribution from the first four of the diagrams in Fig. 1 in a sequential order, we have

$$\begin{aligned} \Gamma_a &= -\frac{1}{\beta^2} \sum_{\omega_1, \omega_2} \int \frac{d^2 \vec{p}_1}{(2\pi)^2} \int \frac{d^2 \vec{p}_2}{(2\pi)^2} \frac{\langle \vec{p}_1 \vec{p}_2 | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle}{(i\omega_1 - p_1^2 + \mu)(i\omega_2 - p_2^2 + \mu)} \\ &= -\beta \int \frac{d^2 \vec{p}_1}{(2\pi)^2} \int \frac{d^2 \vec{p}_2}{(2\pi)^2} \langle \vec{p}_1 \vec{p}_2 | \mathscr{V} | \vec{p}_1 \vec{p}_2 \rangle \, n(\vec{p}_1) \, n(\vec{p}_2) \end{aligned} \tag{31}$$

with

$$n(\vec{p}) = \frac{z e^{-\beta p^2}}{1 - z e^{-\beta p^2}} = \sum_{l=1}^{\infty} z^l e^{-l\beta p^2} \tag{32}$$

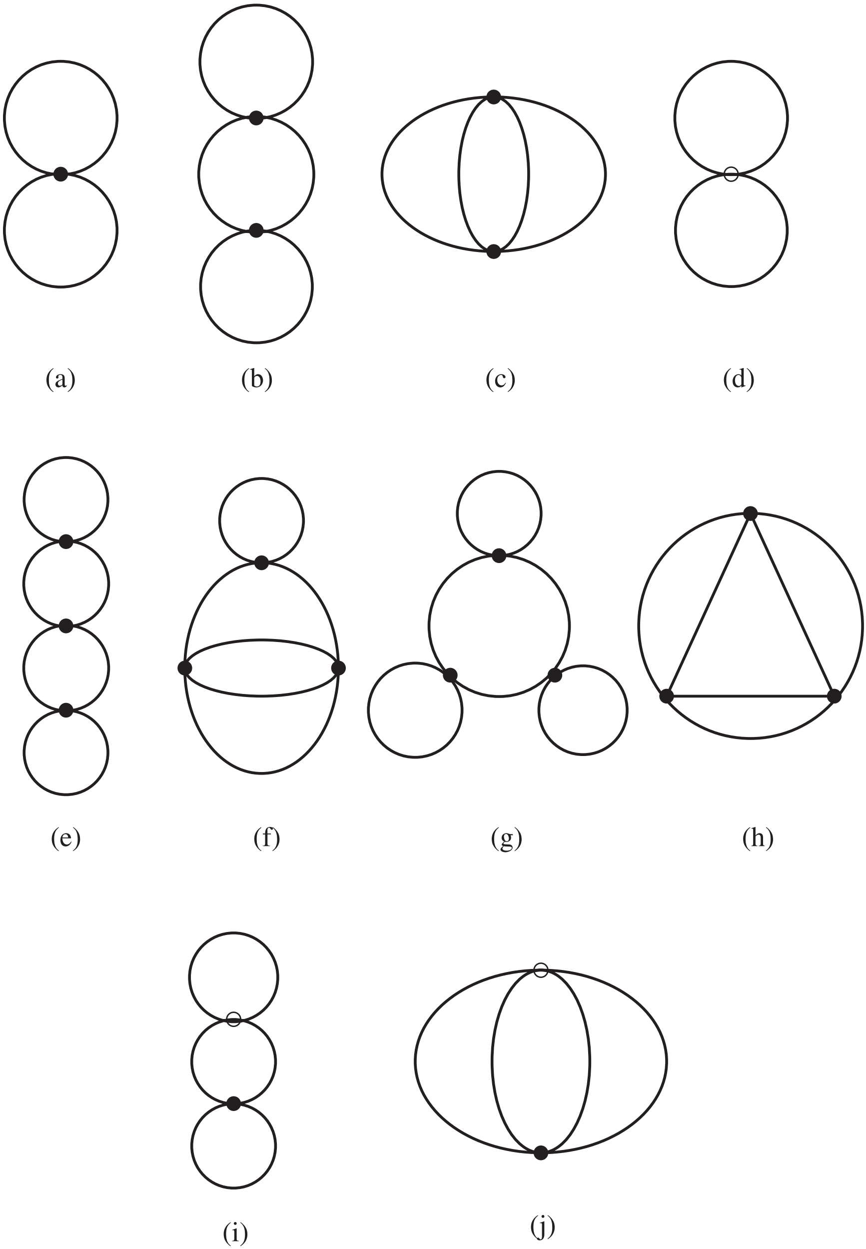

Fig. 1. The thermal diagrams for Virial expansion to the order α^3.

and $z = e^{\beta\mu}$ the fugacity. In accordance with the expansion (25), we have

$$\Gamma_a = \Gamma'_a + \Gamma''_a \tag{33}$$

with

$$\Gamma'_a = -8\pi\alpha\beta \int \frac{d^2\vec{p}_1}{(2\pi)^2} \int \frac{d^2\vec{p}_2}{(2\pi)^2} n(\vec{p}_1)\, n(\vec{p}_2) \left[1 - \alpha\left(\ln \frac{p^2\lambda^2}{2\pi^2} + \gamma\right)\right] \tag{34}$$

and

$$\Gamma''_a = 16\pi\alpha^2\beta \int \frac{d^2\vec{p}_1}{(2\pi)^2} \int \frac{d^2\vec{p}_2}{(2\pi)^2} n(\vec{p}_1)\, n(\vec{p}_2)\, \mathscr{P} \int_0^\infty dk\, k\, \frac{e^{\beta(p^2-k^2)}}{p^2-k^2}. \tag{35}$$

Using Taylor expansion of (32) and integrating over $\vec{p}_1$ and $\vec{p}_2$, we end up with

$$\Gamma'_a = -\frac{2\alpha}{\lambda^2}\left[\ln^2 \frac{1}{1-z} + \alpha D(z) + O(\alpha^2)\right] \tag{36}$$

with

$$D(z) = \sum_{r,\,s=1}^{\infty} \frac{z^{r+s}}{rs} \ln \frac{2rs}{r+s}.$$

Similarly, we have

$$\begin{aligned}
\Gamma_b &= \frac{128\pi^2\alpha^2}{\beta^2} \int \frac{d^2\vec{p}_1}{(2\pi)^2} \int \frac{d^2\vec{p}_2}{(2\pi)^2} \int \frac{d^2\vec{p}_3}{(2\pi)^2} \\
&\quad \times \sum_{\omega_1,\omega_2,\omega_3} \frac{1}{(i\omega_1 - p_1^2 + \mu)^2\, (i\omega_2 - p_1^2 + \mu)(i\omega_3 - p_3^2 + \mu)} \\
&= 128\pi^2\alpha^2\beta^2 \int \frac{d^2\vec{p}_1}{(2\pi)^2} \int \frac{d^2\vec{p}_2}{(2\pi)^2} \int \frac{d^2\vec{p}_3}{(2\pi)^2} n(\vec{p}_1)[n(\vec{p}_1)+1]\, n(\vec{p}_2)\, n(\vec{p}_3) \\
&= \frac{8\alpha^2}{\lambda^2} \frac{z}{(1-z)^2} \ln^2 \frac{1}{1-z}.
\end{aligned} \tag{37}$$

Finally

$$\Gamma_c = 32\pi^2\alpha^2\beta \int \frac{d^2\vec{p}_1}{(2\pi)^2} \int \frac{d^2\vec{p}_2}{(2\pi)^2} \int \frac{d^2\vec{p}_3}{(2\pi)^2} \int \frac{d^2\vec{p}_4}{(2\pi)^2} (2\pi)^2\, \delta^2(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$
$$\times \sum_{\omega_1,\omega_2,\omega_3} \frac{1}{(i\omega_1 - p_1^2+\mu)(i\omega_2 - p_2^2+\mu)(i\omega_3 - p_3^2+\mu)[i(\omega_1+\omega_2-\omega_3)-p_4^2+\mu]}$$
$$= \frac{32\pi^2\alpha^2\beta}{z^2} \int \frac{d^2\vec{p}_1}{(2\pi)^2} \int \frac{d^2\vec{p}_2}{(2\pi)^2} \int \frac{d^2\vec{p}_3}{(2\pi)^2} \int \frac{d^2\vec{p}_4}{(2\pi)^2} (2\pi)^2\, \delta^2(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$
$$\times \frac{e^{\beta(p_1^2+p_2^2)} - e^{\beta(p_3^2+p_4^2)}}{p_1^2+p_2^2-p_3^2-p_4^2}\, n(\vec{p}_1)\, n(\vec{p}_2)\, n(\vec{p}_3)\, n(\vec{p}_4), \tag{38}$$

which represents a genuine three-body scattering, and

$$\Gamma_d = -64\pi^2\alpha^2\beta \int \frac{d^2\vec{p}_1}{(2\pi)^2} \int \frac{d^2\vec{p}_2}{(2\pi)^2} \int \frac{d^2\vec{p}_3}{(2\pi)^2} \int \frac{d^2\vec{p}_4}{(2\pi)^2} (2\pi)^2\, \delta^2(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$
$$\times \frac{e^{\beta(p_1^2+p_2^2-p_3^2-p_4^2)}-1}{p_1^2+p_2^2-p_3^2-p_4^2}\, n(\vec{p}_1)\, n(\vec{p}_2). \tag{39}$$

Things are greatly simplified to the order $O(\alpha^2)$ by forming the following combination

$$\Gamma_c' \equiv \Gamma_a'' + \Gamma_c + \Gamma_d$$
$$= -128\pi^2\alpha^2\beta\mathscr{P} \int \frac{d^2\vec{p}_1}{(2\pi)^2} \int \frac{d^2\vec{p}_2}{(2\pi)^2} \int \frac{d^2\vec{p}_3}{(2\pi)^2} \int \frac{d^2\vec{p}_4}{(2\pi)^2} (2\pi)^2$$
$$\times\, \delta^2(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\, \frac{n(\vec{p}_1)\, n(\vec{p}_2)\, n(\vec{p}_3)}{p_1^2+p_2^2-p_3^2-p_4^2}. \tag{40}$$

This expression is then simplified in four steps:

(1) Expanding $n(\vec{p}_i)$ according to the power of z;

(2) Using the property of the principal value,

$$\mathscr{P}\frac{1}{p_1^2+p_2^2-p_3^2-p_4^2} = \mathrm{Re}\, \frac{1}{p_1^2+p_2^2-p_3^2-p_4^2+i0^+}$$
$$= \mathrm{Im} \int_0^\infty dx\, e^{i(p_1^2+p_2^2-p_3^2-p_4^2+i0^+)\,x} \tag{41}$$

and the Fourier transformation of the delta function,

$$\delta^2(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)=\int \frac{d^2\vec{p}}{(2\pi)^2}\, e^{i(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\cdot\vec{p}}; \tag{42}$$

(3) Carrying out the Gauss integration over $\vec{p}$'s and then the Gauss integration over $\vec{p}$ for each term of the power series in z;

(4) Carrying out the elementary integral over x. The final result reads

$$\Gamma'_c = \frac{4\alpha^2}{\lambda^2} F(z) \tag{43}$$

with

$$F(z) = \sum_{r,s,t=1}^{\infty} \frac{z^{r+s+t}}{\sqrt{rs(r+t)(s+t)}} \ln \frac{\sqrt{(r+t)(s+t)}+\sqrt{rs}}{\sqrt{(r+t)(s+t)}-\sqrt{rs}}.$$

Collecting above results, we obtain the Virial expansion of a dilute Bose gas to the second order of the interaction,

$$\frac{p}{k_B T} = \frac{1}{\lambda^2} g_2(z) - \frac{2\alpha}{\lambda^2} \ln^2 \frac{1}{1-z} + \frac{2\alpha^2}{\lambda^2}\left[\frac{4z}{1-z}\ln^2\frac{1}{1-z} + 2\phi(z)\right] + O(\alpha^3), \tag{44}$$

with

$$\phi(z) \equiv F(z) + \tfrac{1}{2} D(z), \tag{45}$$

and the corresponding number density is given by

$$n = \frac{1}{\lambda^2}\ln\frac{1}{1-z} - \frac{4\alpha}{\lambda^2}\frac{z}{1-z}\ln\frac{1}{1-z} + \frac{2\alpha^2}{\lambda^2}\frac{d}{d\ln z}\left[\frac{4z}{1-z}\ln^2\frac{1}{1-z} + 2\phi(z)\right] + O(\alpha^3). \tag{46}$$

We notice that the order α^2 term start with the third power of z, which may be viewed as a criteria to fix the constant part pertaining the logarithm of the running coupling constant (23). Furthermore, the coefficient of z^2 agrees with the result obtained with the classical formula of Beth and Uhlenbeck.[14] The asymptotic behavior of $D(z)$ and $F(z)$ as $z \to 1^-$ is analyzed in Appendix B.

4. THERMODYNAMICAL FUNCTIONS

Among experimental observables of a two dimensional Bose gas are the homogeneous thermodynamical functions, which follow readily from the Virial expansion of the last section. For the sake of clarity of notations, all thermodynamic functions in their ideal gas limit will carry the superscript "(0)."

Inverting (46), an expression of the fugacity z in terms of the density n is obtained to the order α^2,

$$\ln z = \ln z^{(0)} + 4\alpha\xi - \frac{4\alpha^2}{e^{\xi}-1} \left.\frac{d\phi}{d\ln z}\right|_{z=z^{(0)}} \tag{47}$$

with $\xi \equiv n\lambda^2$ and $z^{(0)} = 1 - e^{-\xi}$, the fugacity of an ideal Bose gas in two dimensions. On substituting (47) back to (44), we obtain the equation of state to the order α^2,

$$p = p^{(0)} + 8\pi\alpha n^2 + \frac{16\pi}{\lambda^4}\alpha^2 \left[\phi(1-e^{-\xi}) - \frac{\xi}{e^{\xi}-1} \left.\frac{d\phi}{d\ln z}\right|_{z=1-e^{-\xi}} \right], \tag{48}$$

where

$$p^{(0)} = \frac{4\pi}{\lambda^4} g_2(1-e^{-\xi}) = \frac{4\pi}{\lambda^4}\left[\xi - \frac{\xi^2}{4} + \sum_{m=1}^{\infty} \frac{(-)^{m-1} B_m}{(2m+1)!} \xi^{2m+1} \right] \tag{49}$$

is the equation of state for an ideal gas with B_m the mth Bernoullian number.

The Helmholtz free energy per unit volume is obtained through the formula

$$f = \mu n - p = f^{(0)} + 8\pi\alpha n^2 - \frac{16\pi\alpha^2}{\lambda^4} \phi(1-e^{-\xi}). \tag{50}$$

The entropy per unit volume is given by

$$s = \left(\frac{\partial p}{\partial T}\right)_{\mu} = s^{(0)} - \frac{16\pi\alpha^2}{\lambda^4 T}\left[\xi^2 - 2\phi(1-e^{-\xi}) + \frac{\xi}{e^{\xi}-1} \left.\frac{d\phi}{d\ln z}\right|_{z=1-e^{-\xi}} \right] \tag{51}$$

and the correction to the ideal limit is of the α^2. It follows from (50) and (51) that the internal energy per unit volume is

$$u = f + Ts = u^{(0)} + 8\pi\alpha n^2 - \frac{16\pi\alpha^2}{\lambda^4}\left[\xi^2 - \phi(1-e^{-\xi}) + \frac{\xi}{e^{\xi}-1} \left.\frac{d\phi}{d\ln z}\right|_{z=1-e^{-\xi}} \right]. \tag{52}$$

The specific heat at a constant volume(area) reads

$$c_V = T\left(\frac{\partial s}{\partial T}\right)_n$$
$$= c_V^{(0)} + \frac{4\alpha^2 k_B}{\lambda^2}\left[\xi^2 + 2\phi(z) - \frac{\xi(2e^{\xi}-2+\xi e^{\xi})}{(e^{\xi}-1)^2}\frac{d\phi}{d\ln z} + \frac{\xi^2}{(e^{\xi}-1)^2}\frac{d^2\phi}{d(\ln z)^2}\right]\Bigg|_{z=1-e^{-\xi}}, \tag{53}$$

and the leading order contribution of the interaction is proportional to α^2. The isothermal compressibility, κ_T can be calculated readily from (48) with the results

$$\kappa_T = -\Omega\left(\frac{\partial p}{\partial \Omega}\right)_T = \kappa_T^{(0)} + 16\pi\alpha n^2 + \frac{16\pi\alpha^2 n^2}{(e^{\xi}-1)^2}\left[e^{\xi}\frac{d\phi}{d\ln z} - \frac{d^2\phi}{d(\ln z)^2}\right]\Bigg|_{z=1-e^{-\xi}}. \tag{54}$$

Employing the thermodynamic relationship

$$\kappa_S - \kappa_T = -\frac{n^2 T}{c_V}\left(\frac{\partial p}{\partial T}\right)_n\left[\frac{\partial}{\partial n}\left(\frac{s}{n}\right)\right]_T \tag{55}$$

we find the adiabatic compressibility

$$\kappa_S = -\Omega\left(\frac{\partial p}{\partial \Omega}\right)_S$$
$$= \kappa_S^{(0)} + 16\pi\alpha n^2 + 16\pi\alpha^2 n^2 + \frac{32\pi\alpha^2}{\lambda^4}\left[\phi(z) - \frac{\xi}{e^{\xi}-1}\frac{d\phi}{d\ln z}\right]\Bigg|_{z=1-e^{-\xi}}, \tag{56}$$

which can be directly measured through the sound speed in the Bose gas,

$$v = \sqrt{\frac{2\kappa_S}{n}}. \tag{57}$$

5. CONCLUDING REMARKS

In previous sections, we have developed a diagrammatic approach to the Virial expansion of a dilute Bose gas with a repulsive interaction of a short range, valid in the region where $n\lambda^2 \sim 1$ and $\lambda \gg a$. Before concluding the paper, we shall remark on its implication on the quasi Bose–Einstein condensation and its applicability to the realistic system of an quasi-2D gas of alkaline atoms in a strongly anisotropic trap.

All coefficients of the Virial expansion (44) are singular at $z = 1$, and the singularity gets stronger at higher orders. A simple power counting argument shows that the coefficient of α^N diverges as $(1-z)^{1-N}$ as $z \to 1^-$, up to some powers of $\ln \frac{1}{1-z}$, as is bore out in the explicit form of (44) to α^2 (see Appendix B for the singularities of $D(z)$ and $F(z)$). Therefore the reliability of the expansion requires that

$$\alpha \ll 1 - z. \tag{58}$$

Using formula (23) for α and the ideal gas limit of z, condition (58) becomes

$$T \gg \frac{4\pi n}{\ln \ln \frac{1}{2\pi n a^2}}, \tag{59}$$

which is consistent with the formula of the transition temperature to the superfluid phase, obtained in refs. 6 and 8, under a mean field approximation. It will be interesting to see whether the resummation of diagrams developed in ref. 16 for a 3D hard sphere Bose gas can be applied here to extract the ground state energy beyond the mean-field approximation and to compare with the results in refs. 7 and 9. At this stage, we only remark on that the order α correction to the internal energy is twice of the rigorous result of Lieb and Yngvason[17] for the ground state energy for the thermal wavelength comparable with the inter-boson distance. The factor two comes from the exchange energy which is absent at zero temperature because of the Bose condensate. For a point-like repulsion corresponding to the leading order term of (25), the exchange energy is equal to the classical energy and thereby doubles the interaction energy under the mean-field approximation. A similar effect was observed in three dimensions.[16] [2]

The Virial expansion in two dimensions is characterized by the logarithm in the denominator of the running coupling constant (23) with a universal coefficient. The observation of such a universality demands the logarithm to be large enough a large enough such that the first few terms of the expansion represent a reasonable approximation. Experimentally,

[2] Another way to understand the factor two is to consider the expectation value of the potential energy of a point-like repulsion with respect to a state of $\mathcal{N}(=n\Omega)$ free bosons, i.e., $E \equiv \langle | \frac{V}{2\Omega} \sum_{\vec{p}_1,\vec{p}_2,\vec{p}_1',\vec{p}_2'} \delta_{\vec{p}_1+\vec{p}_2,\vec{p}_1'+\vec{p}_2'} b^\dagger_{\vec{p}_2'} b^\dagger_{\vec{p}_1'} b_{\vec{p}_1} b_{\vec{p}_2} | \rangle$ with $|\rangle$ a product of single particle states. The summation can be broken into three nonvanishing terms: $E = \frac{V}{2\Omega}[\sum_{\vec{p}_1,\vec{p}_2} \langle | b^\dagger_{\vec{p}_2} b^\dagger_{\vec{p}_1} b_{\vec{p}_1} b_{\vec{p}_2} | \rangle + \sum_{\vec{p}_1,\vec{p}_2} \langle | b^\dagger_{\vec{p}_1} b^\dagger_{\vec{p}_2} b_{\vec{p}_1} b_{\vec{p}_2} | \rangle - \sum_{\vec{p}} \langle | b^\dagger_{\vec{p}} b^\dagger_{\vec{p}} b_{\vec{p}} b_{\vec{p}} | \rangle]$. In the limit of an infinite volume, $\Omega \to \infty$, each of the first and second terms gives rise to $\mathcal{N}^2$. If all bosons occupy one momentum level (Bose condensate), the third term contributes $-\mathcal{N}^2$, making the sum $\frac{E_0}{\Omega} = \frac{1}{2} V n^2$. If none of the levels are macroscopically occupied, the third term is $O(\mathcal{N})$ and will not contribute to the infinite volume limit. We have then $\frac{E_0}{\Omega} = V n^2$.

a quasi-two dimensional gas can be implemented with a strongly anisotropic trap, which can be modeled as a three dimensional gas in a narrow harmonic well in one direction, referred to as the trapped direction. An analytical solution to the two-body scattering for large extension of the single particle wave function in the trapped direction, l, in comparison with the range of the inter-particle interaction (which remains three-dimensional) was obtained in ref. 18, and an effective phase shift of the two dimensional s-wave component can be extracted when the de Broglie wavelength in the trapped plane becomes much longer than l,

$$\delta_{\text{eff.}} = -\frac{\pi}{2\ln\frac{1}{ka_{\text{eff.}}}}, \tag{60}$$

where k is the relative 2D momentum and $a_{\text{eff.}} = \sqrt{\pi}\, l \exp(-\sqrt{\frac{\pi}{2}}\frac{l}{a})$ with a the 3D scattering length. The authors of ref. 18 also found numerically that the approximation (60) works well even when $l \sim a$. The typical inter-particle distance in an atomic trap is about 10^4 Å and the typical 3D scattering length for alkaline atoms is of the order of 100 Å. For the thermal wavelength equal to the inter-particle distance and the trapped dimension equal to a (which is technically feasible), we have $a_{\text{eff.}} \simeq 51$ Å and then $\alpha \simeq 0.12$, according to (23) with a there replaced by $a_{\text{eff.}}$. The universal 2D logarithm could be quite significant to observations.

APPENDIX A

To regularize the infinite volume limit of $V_m(p', p)$ defined in (14), we follow the usual practice by restricting the relative coordinate $\vec{r}$ within a large circle of radius R and imposing the Dirichlet boundary conditions for the wave functions with and without potential V, i.e.,

$$u_m(k \mid R) = 0 \tag{A1}$$

and

$$u_m^{(0)}(p \mid R) = 0, \tag{A2}$$

where $u_m(k \mid r)$ is the solution of the radial Schrödinger equation, regular at $r = 0$,

$$\left[-\frac{2}{r}\frac{d}{dr}(r\frac{d}{dr}) + \frac{2m^2}{r^2} - V(r)\right] u_m(k \mid r) = 2k^2 u_m(k \mid r), \tag{A3}$$

and $u_m^{(0)}(p\,|\,r) = J_m(pr)$ satisfies the radial Schrödinger equation of a free particle, i.e.,

$$\left[-\frac{2}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)+\frac{2m^2}{r^2}\right]u_m^{(0)}(k\,|\,r) = 2p^2 u_m^{(0)}(k\,|\,r). \tag{A4}$$

For $pr \gg 1$, the asymptotic behavior of $u_m^{(0)}(p\,|\,r)$ is

$$u_m^{(0)}(p\,|\,r) \simeq \sqrt{\frac{2}{\pi pr}}\cos\left(pr - \frac{m\pi}{2} - \frac{\pi}{4}\right) \tag{A5}$$

and that of $u_m(k\,|\,r)$ for $kr \gg 1$ is given by (12). It follows from (A1), (A2), (A5), and (12) that

$$p = p_n = \left(n + \frac{m}{2} + \frac{1}{4}\right)\frac{\pi}{R} \tag{A6}$$

and

$$k = k_n = p_n - \frac{\delta_m(p_n)}{R} \tag{A7}$$

for large R.

Multiplying (A3) by $J_m(pr)$ and (A2) by $u_m(k\,|\,r)$, subtracting the result and using the boundary condition (A1) and (A2), we find

$$\int_0^R dr\, rJ_m(pr)\, u_m(k\,|\,r) = \frac{v_m(p, k)}{2(k^2 - p^2)}, \tag{A8}$$

where

$$v_m(p, k) = \int_0^R dr\, rJ_m(pr)\, V(r)\, u_m(k\,|\,r), \tag{A9}$$

and is well behaved in the limit $R \to \infty$. Equation (14) becomes then

$$V_m(p', p) = \pi \sum_k N_{km}^2 e^{-2\beta k^2} \frac{v_m(p', k)\, v_m(p, k)}{k^2 - p^2} \tag{A10}$$

with N_{km} the normalization constant such that

$$N_{km}^2 \int_0^R dr\, r u_m^2(k\,|\,r) = 1. \tag{A11}$$

Using the asymptotic behavior (12), we find $N_{km} \simeq \sqrt{\frac{\pi k}{R}}$ for large R.

To take the limit $R \to \infty$ of the sum (A10), we need to isolate out the k's which are sufficiently close to p. For this purpose, we introduce a subset of k's, B, such that all $k \in B$ satisfy the condition that $|k-p| < \frac{N\pi}{R}$ for $N \gg 1$. We further specify the order of the limit such that $R \to \infty$ first and then $N \to \infty$. Consequently, the summation in (A10) is divided into two parts,

$$V_m(p', p) = V_m^<(p', p) + V_m^>(p', p), \tag{A12}$$

with $V_m^<(p', p)$ including only the k's within B and $V_m^<(p', p)$ all others. For $V_m^<(p', p)$, all the k's except those in the denominator can be set to p, and we obtain that

$$\begin{aligned} V_m^<(p', p) &\simeq \frac{\pi^2}{2} e^{-2\beta p^2} v_m(p', p)\, v_m(p, p) \sum_{l=-N}^{N} \frac{1}{\pi l - \delta_m(p)} \\ &= \frac{\pi^2}{2} e^{-2\beta p^2} v_m(p', p)\, v_m(p, p) \cot \delta_m(p). \end{aligned} \tag{A13}$$

The summation $V_m(p', p)$, however can be replaced simply by the principal value of an integral, i.e.

$$V_m^>(p', p) = \pi \mathscr{P} \int_0^\infty dk\, k e^{-2\beta k^2} \frac{v_m(p', k)\, v_m(p, k)}{k^2 - p^2}. \tag{A14}$$

Following the same steps that leads to (A8), we derive that

$$v_m(p, p) = -\frac{4}{\pi} \sin \delta_m(p). \tag{A15}$$

APPENDIX B

In this appendix, we shall determine the singular behavior of the functions $D(z)$ and $F(z)$ in the Virial expansion (44) as $z \to 1^-$.

On writing $D(z) = D_1(z) + D_2(z)$ with

$$D_1(z) \equiv \sum_{r,s=1}^{\infty} \frac{z^{r+s}}{rs} \ln 2rs \tag{B1}$$

and

$$D_2(z) \equiv -\sum_{r,s=1}^{\infty} \frac{z^{r+s}}{rs} \ln(r+s), \tag{B2}$$

we have

$$D_1(z) = 2S_1(z)\, S_2(z) + S_1^2(z) \ln 2, \tag{B3}$$

where

$$S_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = \ln \frac{1}{1-z} \tag{B4}$$

and

$$S_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \ln n \simeq \frac{1}{2} \ln^2 \frac{1}{1-z}, \tag{B5}$$

as $z \to 1^-$. To estimate $D_2(z)$, we rewrite it as

$$D_2(z) = \sum_{N=1}^{\infty} C_N z^N \tag{B6}$$

where

$$C_N = \ln N \sum_{m+n=N} \frac{1}{mn} = \frac{2}{N} \ln N \sum_{n=1}^{N-1} \frac{1}{n} \simeq \frac{2}{N} \ln^2 N \tag{B7}$$

for $N \gg 1$. It follows then that

$$D_2(z) \simeq 2 \sum_{N=1}^{\infty} \frac{z^N}{N} \ln^2 N \simeq \frac{2}{3} \ln^3 \frac{1}{1-z}. \tag{B8}$$

Combining (B1) and (B2) with the aid of (B3)–(B5) and (B8), we obtain the asymptotic formula

$$D(z) \simeq \frac{1}{3} \ln^3 \frac{1}{1-z}. \tag{B9}$$

The asymptotic behavior of $F(z)$ can be determined by its integral representation, read off from (40) and (43)

$$F(z) = -128\pi^3\beta^2\mathscr{P}\int\frac{d^2\vec{p}_1}{(2\pi)^2}\int\frac{d^2\vec{p}_2}{(2\pi)^2}\int\frac{d^2\vec{p}_3}{(2\pi)^2}\int\frac{d^2\vec{p}_4}{(2\pi)^2}(2\pi)^2$$
$$\times\delta^2(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{n(\vec{p}_1)\,n(\vec{p}_2)\,n(\vec{p}_3)}{p_1^2+p_2^2-p_3^2-p_4^2}$$
$$= -32\pi^3\beta^2\int\frac{d^2\vec{p}_1}{(2\pi)^2}\int\frac{d^2\vec{p}_2}{(2\pi)^2}\int\frac{d^2\vec{p}_3}{(2\pi)^2}\int\frac{d^2\vec{p}_4}{(2\pi)^2}(2\pi)^2\,\delta^2(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$$
$$\times\frac{n(\vec{p}_1)\,n(\vec{p}_2)[n(\vec{p}_3)+n(\vec{p}_4)]-n(\vec{p}_3)\,n(\vec{p}_4)[n(\vec{p}_1)+n(\vec{p}_2)]}{p_1^2+p_2^2-p_3^2-p_4^2}. \quad \text{(B10)}$$

On writing $z=e^{-\epsilon}$, the leading singularity of $F(z)$ in the limit $z\to 1^-$ can be extracted from the integration domain where $p_j<\frac{\eta}{\sqrt{\beta}}$ for $j=1,2,3,4$ with $\epsilon\ll\eta\ll 1$. We have then

$$F(z)\simeq 32\pi^3\int_{k_j<\eta}\prod_{j=1}^{4}\frac{d^2\vec{k}_j}{(2\pi)^2}\frac{(2\pi)^2\,\delta^2(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)}{(k_1^2+\epsilon)(k_2^2+\epsilon)(k_3^2+\epsilon)(k_4^2+\epsilon)}$$
$$\simeq\frac{4}{\epsilon}\int_0^\infty dx\,xK_0^4(x) \quad \text{(B11)}$$

with $K_0(x)$ the modified Bessel function of the second kind.

ACKNOWLEDGMENTS

I would like to thank Professor N. Khuri for valuable discussions. This work is supported in part by the US Department of Energy under Grants DE-FG02-91ER40651-TASKB.

REFERENCES

1. F. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Rev. Mod. Phys.* **71**:463 (1999).
2. T. D. Lee and C. N. Yang, *Phys. Rev.* **105**:1119 (1957).
3. T. D. Lee and C. N. Yang, *Phys. Rev.* **113**:1165 (1959); **116**:25 (1959).
4. E. Lieb, *J. Math. Phys.* **8**:43 (1967).
5. M. Schick, *Phys. Rev. A* **3**:1067 (1971).
6. V. N. Popov, *Theor. and Math. Phys.* **11**:565 (1977).
7. D. F. Hines, N. E. Frankel, and D. J. Mitchell, *Phys. Lett. A* **3**:1067 (1978)
8. D. S. Fisher and P. C. Hohenberg, *Phys. Rev. B* **37**:4936 (1988).
9. A. A. Ovchinnikov, *J. Phys. Condens. Mat.* **5**:8665 (1993).
10. J. O. Anderson, *Eur. Phys. J. B* **28**:389 (2002).
11. S. K. Adhikari, *Amer. J. Phys.* **54**:362 (1986).
12. K. Chadan, N. N. Khuri, A. Martin, and T. T. Wu, *Phys. Rev. D* **58**:025014 (1998).
13. P. C. Hohenberg, *Phys. Rev.* **158**:383 (1967).

14. R. Friedberg, T. D. Lee, and Hai-cang Ren, *Phys. Rev. B* **50**:10190 (1994).
15. E. Beth and G. E. Uhlenbeck, *Physica* **4**:915 (1937).
16. T. D. Lee and C. N. Yang, *Phys. Rev.* **117**:12 (1960).
17. E. H. Lieb and J. Yngvason, *J. Stat. Phys.* **103**:509 (2001).
18. D. S. Petrov, M. Holzmann, and G. V. Shlyapnikov, *Phys. Rev. Lett.* **84**:2551 (2000).